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NONSTATIONARY IDEAL INCOMPRESSIBLE FLUID FLOWS: CONDITIONS OF EXISTENCE AND UNIQUENESS OF SOLUTIONS

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The problem of formulating minimal conditions on input data that can guarantee the existence and uniqueness of solutions of the boundary value problems describing non-one-dimensional ideal incompressible fluid flow is considered using as an example the initial boundary value problem in a space–time cylinder constructed on a bounded flow domain with the nonpenetration condition on its boundary (which corresponds to fluid flow in a closed vessel). The existence problems are considered only for plane flows, and the uniqueness issues for three-dimensional flows as well. The required conditions are obtained in the form of conditions specifying that the vorticity belongs to definite functional Orlicz spaces. The results are compared with well-known results. Examples are given of admissible types of singularities for which the obtained results are valid, which is a physical interpretation of these results.

Key words: *Euler equations, ideal incompressible fluid, nonstationary flows, generalized solutions, Orlicz spaces, Gronwall lemma.*

Introduction. As is known, ideal incompressible fluid flow is described by the Euler equations [1]

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + \nabla p = \boldsymbol{f}; \tag{1}$$

$$\operatorname{div} \boldsymbol{v} = 0, \tag{2}$$

where $\boldsymbol{v} = (v_1, \ldots, v_n)$ is the velocity, p is the pressure, and $\boldsymbol{f}$ is the specified vector of external mass forces. The quantities $\boldsymbol{v}$, p, and $\boldsymbol{f}$ are functions of time t and the spatial variables $\boldsymbol{x} = (x_1, \ldots, x_n)$, where $n = 3$ corresponds to three-dimensional flows of the general type, and $n = 2$ to plane–parallel flows.

The present paper deals with the problem of the mathematical well-posedness of model (1), (2), i.e., the question of the existence and uniqueness of solutions of boundary value problems for this system. This problem is classical; a fairly complete review of the results obtained in its solution is given in [2, 3]. Among the papers on the well-posedness of model (1), (2), special mention should be made of the classical papers of Kato [4], Yudovich [5, 6], and Kazhikhov [7], which underlie research in this area. In [6, 7], the physically more interesting problem of fluid flow through a specified domain was considered, and in [4, 5], foundations are laid for the global theory of existence and uniqueness of solutions of the initial boundary value problems for model (1), (2), i.e., corresponding global theorems are obtained (for arbitrary large values of time and initial data). This was done for a two-dimensional problem with initial boundary value conditions of the form

$$\boldsymbol{v}\Big|_{t=0} = \boldsymbol{v}_0; \tag{3}$$

$$\boldsymbol{v} \cdot \boldsymbol{n}\Big|_{\partial\Omega \times (0,T)} = 0, \tag{4}$$

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where Ω is a bounded domain in $\mathbb{R}^2$ with the boundary belonging to the class C^2 (or $C^{2+\alpha}$ in [4]), $\boldsymbol{n}$ is the outward normal to $\partial\Omega$, $T > 0$ is an arbitrary time interval, and $\boldsymbol{v}_0$ is the initial velocity. The solution of problem (1)–(4) is sought in the space–time cylinder $Q_T = \Omega \times (0, T)$. The initial velocity should satisfy conditions (2) and (4). Problem (1)–(4) describes the motion of a fluid in a closed vessel and is not very interesting from a practical point of view. However, it is reasonable to begin the mathematical investigation with this problem as a model one and then to extend it to other initial boundary value problems. For three-dimensional motion, the problem of the global mathematical well-posedness of model (1), (2) has proved much more difficult. In particular, the question of the global existence of solutions has not been solved as yet.

In the present work, we also confine ourselves to investigation of problem (1)–(4). The existence problem is considered only for the case $n = 2$. The goal of the study is to refine the conditions on the input data and on the desired solution that guarantee the global existence and uniqueness of solutions. In particular, it is of interest to prove the existence of a solution for nonsmooth input (especially initial) data and to ensure the uniqueness of the solution under the weakest (as far as possible) constraints on it. Thus, we will not consider the classical solutions of the problem constructed in [4], but we will examine the generalized solutions for which theoretical foundations were laid in [5]. In this case, it is sufficient to consider the case of a simply-connected bounded domain Ω because extension to other domains can be performed in the same way as in [5]. In [5], the existence of a solution of problem (1)–(4) was proved for bounded initial vorticity [$\operatorname{rot} \boldsymbol{v}_0 \in L_\infty(\Omega)$], and the solution itself belonged to the following class:

$$\operatorname{rot} \boldsymbol{v} \in L_\infty(Q_T), \qquad \|\nabla p\|_{L_\infty(0,T,L_r(\Omega))} \leqslant Cr \quad \forall r \gg 1. \tag{5}$$

It has also been shown [5] that, in class (5), the solution of the problem is unique. Among the results that have been added to the indicated result [5] in the last 45 years, mention should be made of the following.

1. Morgulis [8] proved the existence of a solution of problem (1)–(4) for $n = 2$ with a rather "bad" initial velocity:

$$\operatorname{rot} \boldsymbol{v}_0 \in L_M(\Omega). \tag{6}$$

Here $L_M(\Omega)$ is an Orlicz space generated by any N-function of M such that its complementary N-function $\bar{M}$ satisfies the condition

$$\int^{+\infty} \bar{M}'(t) t^{-2} \exp(-t^2/\gamma)\, dt < \infty \tag{7}$$

with a certain constant $\gamma(\Omega)$ (the main concepts related to Orlicz spaces are given in Sec. 1). Conditions (6) and (7) imply that, although $\operatorname{rot} \boldsymbol{v}_0$ belongs to the space $L_1(\Omega)$, it does not belong to $L_r(\Omega)$ for any $r > 1$.

2. Yudovich [2] proved the uniqueness of solution of (1)–(4) for weaker constraints on the solution, namely for

$$\|\operatorname{rot} \boldsymbol{v}\|_{L_\infty(0,T,L_r(\Omega))} \leqslant C\theta(r) \qquad (r \gg 1) \tag{8}$$

in the case of fairly slowly (logarithmically) increasing functions θ. This result is based on an analysis of the Gronwall type inequality

$$\int_\Omega \psi(t, \boldsymbol{x})\, d\boldsymbol{x} \leqslant \int_0^t \int_\Omega g(s, \boldsymbol{x}) \psi(s, \boldsymbol{x})\, d\boldsymbol{x}\, ds \tag{9}$$

with nonnegative functions ψ and g (for a specified function g, it is required to prove that $\psi = 0$); for bounded g, the analysis of (9) is trivial (reduces to the application of the classical Gronwall lemma). For g satisfying an estimate of the form (8) with $\theta(r) = r$, this analysis was performed in [5] and was then strengthened in [2].

It should be noted, however, that conditions (6)–(8) are insufficiently constructive. In particular, condition (8) is a family of conditions; in addition, it contains too awkward constraints on θ, which are difficult to verify. With all that, the question of the optimality of these results remains open.

In the present work, instead of (6)–(8), more clearly formulated conditions on the initial data and solution are obtained to prove the global existence and uniqueness theorems, and condition (8) is replaced by the more natural and easily verified condition that the vorticity belongs to special Orlicz spaces. This is done, in particular, using the results of [9] for inequality (9). It should be noted that, in [2, 9], the relationship between inequality (9) and the uniqueness problem for fluid particle trajectories was found independently but investigation of this relationship was performed in different terms: by using spaces of the form (8) in [2] and by using Orlicz spaces in [9]. The second method seems to be more natural (in particular, because vorticity is assumed to belong to only one space and not to a family) for both the Euler equations and the problem of trajectories and the problem of inequality (9).

To formulate the results obtained, we need some data from analysis and auxiliary constructions (see Sec. 1). In Sec. 2, conditions are formulated that ensure the unique solvability of the two-dimensional problem (1)–(4) in classes in which (for any n) the solution is unique and the results are compared with the similar results obtained in [2]. The purpose of Sec. 3 is to find the most general conditions (on the input data) that guarantee the solvability of the two-dimensional problem and to give a physical interpretation of the results obtained.

1. Auxiliary Data and Constructions. As noted above, the formulated problem can be studied using functional Orlicz spaces. The theory of these spaces is described in detail in [10]. In the present paper, the necessary information on them is given only briefly.

A function M of one real variable is called an N-function if it is convex and even (hence, one can examine its behavior only on the right half-axis) and if, on the right half-axis, it increases strictly and satisfies the relations

$$\frac{M(s)}{s} \to 0 \quad \text{at} \quad s \to 0, \qquad \frac{M(s)}{s} \to +\infty \quad \text{at} \quad s \to +\infty.$$

For any N-function M, it is possible to determine its complementary N-function $\bar{M}$ which is its Legendre transformation. Since any N-function is differentiable almost everywhere, $\bar{M}$ can be defined as $\bar{M}' = (M')^{-1}$.

If the set $\Omega \subset \mathbb{R}^n$ has a finite measure (for example, is a bounded domain which will further be considered), the Orlicz class $K_M(\Omega)$ can be introduced as a set of measurable functions u such that $\int\limits_\Omega M(u(\boldsymbol{x}))\,d\boldsymbol{x} < \infty$. The Orlicz space $L_M(\Omega)$ is the linear span of the class $K_M(\Omega)$; therefore, it is natural to introduce the Luxemburg norm in it:

$$\|u\|_{L_M(\Omega)} = \inf\Big\{k \;\Big|\; \int\limits_\Omega M\Big(\frac{u(\boldsymbol{x})}{k}\Big)\,d\boldsymbol{x} \leqslant 1\Big\}.$$

Since the measure of Ω is finite, only the behavior of N-functions on $+\infty$ is important; therefore, below it is assumed that the formulas for them are written for large values of the arguments.

Example 1.1. $M(s) = s^p/p$, $p > 1$, $\bar{M}(s) = s^q/q$, $q = p/(p-1)$, $L_M(\Omega) = L_p(\Omega)$, and $L_{\bar{M}}(\Omega) = L_q(\Omega)$.

Example 1.2. $M(s) = \mathrm{e}^s - s - 1$ and $\bar{M}(s) = (s+1)\ln(s+1) - s$. The space $L_M(\Omega)$ consists of functions which belong to all $L_r(\Omega)$, $r < \infty$ (but are nevertheless unbounded, generally speaking), and the space $L_{\bar{M}}(\Omega)$ consists of functions which are integrable (and even possess somewhat better properties) but do not belong to any $L_{1+\varepsilon}(\Omega)$.

Smooth functions, generally speaking, are not dense in $L_M(\Omega)$; therefore, their closure $E_M(\Omega)$ [in the norm $L_M(\Omega)$] generally forms a separable subspace in $L_M(\Omega)$, and $L_{\bar{M}}(\Omega) = (E_M(\Omega))^*$, so that the bounded sets in $L_M(\Omega)$ are weak-star sequentially compact. The spaces $E_M(\Omega)$ and $L_M(\Omega)$ coincide only in the case where M satisfies the Δ_2-condition: $M(2u) \leqslant CM(u)$ for $u \gg 1$, which that, at infinity, the growth of M is no faster than polynomial. This condition is also a criterion of the coincidence of the sets $K_M(\Omega)$ and $L_M(\Omega)$.

An increase in the N-functions at infinity can be compared by means of the relations "$\prec$" and "$\prec\!\!\!\prec$" defined as follows:

$$M_1 \prec M_2 \qquad \text{if} \quad M_1(u) \leqslant M_2(Cu), \qquad u \gg 1,$$

$$M_1 \prec\!\!\!\prec M_2 \qquad \text{if} \quad M_2(u)/M_1(Cu) \to \infty, \qquad u \to \infty \quad \forall C > 0.$$

In the first case, there is the continuous embedding $L_{M_2}(\Omega) \hookrightarrow L_{M_1}(\Omega)$, and, in the second case, this embedding is strict (in the theoretical-set sense) and, in some sense, compact [for example, $L_{M_2}(\Omega) \subset K_{M_1}(\Omega)$]. Accordingly, the relation $M_1 \sim M_2$ (understood as the simultaneous satisfaction of the relations $M_1 \prec M_2$ and $M_2 \prec M_1$) is a criterion of the coincidence $L_{M_1}(\Omega) = L_{M_2}(\Omega)$.

For the functions with growth rate faster than polynomial, the so-called Δ^2-condition can be considered, which implies that $M^2 \sim M$, i.e., $M^2(u) \leqslant M(Cu)$ for $u \gg 1$. Ignoring "pathological" cases (which do not arise in the given work), this condition is satisfied by all N-functions M that increase not more slowly than functions of the form $F(s) = \exp(s^\varepsilon)$.

Thus, the Lebesgue spaces $L_r(\Omega)$ are a particular case of the Orlicz spaces, for which the theory is partially similar to the theory of the spaces L_r (especially in the case where M and $\bar{M}$ simultaneously satisfy the Δ_2-condition) but differs from it in many respects and allows a description of the subtle properties of the functions in Ω.

Between the Lebesgue and Orlicz spaces there is also the following relationship. We consider the set of measurable functions u belonging to all $L_p(\Omega)$ for $p \in [\alpha, \beta)$ such that $\|u\|_{L_p(\Omega)} \leqslant C\omega(p)$, where $p \in [\alpha, \beta)$, with the specified function ω. As has been noted in various cases by various researchers (see the review in [11]), this set is contained in the Orlicz space $L_M(\Omega)$ with an appropriate function M. A fairly full and systematic study of this relationship is made in [11, 12]. The present work uses the terminology and some results of [11]. Below, we consider only the case $\beta = +\infty$. The set described above becomes a Banach space $L_{\omega,\infty}$ if it is endowed with the norm

$$\|u\|_{L_{\omega,\infty}} = \sup_{p\in[\alpha,+\infty)} \frac{\|u\|_{L_p(\Omega)}}{\omega(p)}. \tag{1.1}$$

This space does not depend on the choice of α (i.e., the corresponding norms are equivalent) and does not vary if ω varies to within equivalence of the special form:

$$\omega_1 \overset{\mathcal{L}}{\sim} \omega_2 \qquad \Longleftrightarrow \qquad C_1\omega_1(p) \leqslant \omega_2(p) \leqslant C_2\omega_1(p). \tag{1.2}$$

The operators

$$\mathbf{In}_\infty[\omega](v) = \int\limits_\alpha^{+\infty} \frac{v^p\,dp}{\omega^p(p)}, \qquad \mathbf{Sc}_\infty[\Phi](p) = \max_{v\geqslant 1} \frac{v}{\Phi^{1/p}(v)}$$

implement correspondence between the functions $\omega = \omega(p)$ and the N-functions $\Phi = \Phi(v)$ (see [11]), namely, in the case of rapidly increasing Φ, i.e., slowly increasing ω (which are used below), the indicated correspondence becomes the equality

$$L_{\omega,\infty} = L_\Phi \qquad \text{at} \quad \Phi = \mathbf{In}_\infty[\omega], \quad \text{i.e.,} \quad \omega = \mathbf{Sc}_\infty[\Phi].$$

Conversely, any space L_Φ with a rapidly increasing Φ can be represented as $L_{\omega,\infty}$ with the corresponding function ω.

Thus, in the case considered, the family of estimates in L_p is equivalent to the estimate in the corresponding Orlicz space. Using this fact, one can attempt to convert the results of [2] in terms of Orlicz spaces (see Sec. 2). However, for inequality (9), this conversion is unreasonable because, in [9], inequality (9) was studied directly in terms of Orlicz spaces, and the result was shown to be unimprovable. A partial formulation of this result is given below.

Definition 1.1. The class $\mathcal{K}$ is a set of N-functions that satisfy one of the three equivalent constraints:

$$\int^{+\infty} \frac{\ln M(s)}{s^2}\,ds = +\infty, \qquad \int^{+\infty} \frac{ds}{\bar{M}(s)} = +\infty, \qquad \int^{+\infty} \frac{ds}{sM^{-1}(s)} = +\infty. \tag{1.3}$$

Remark 1.1. All elements $\mathcal{K}$, except in "pathological" cases, satisfy the Δ^2-condition.

Statement 1.1. *Let functions g and ψ be specified in Q_T and nonnegative, $\psi \in L_{1+\varepsilon}(Q_T)$, $g \in K_M(Q_T)$, $M \in \mathcal{K}$. Then, relation* (9) *implies that $\psi = 0$. If $M \notin \mathcal{K}$, there exist nonnegative $\psi \in L_\infty(Q_T)$ and $g \in L_\infty(0, T, L_M(\Omega))$ such that inequality* (9) *is satisfied in spite of $\psi \not\equiv 0$.*

From (1.3) it follows that the class $\mathcal{K}$ consists of functions growing faster than all polynomials. For any $M \in \mathcal{K}$, it is possible to find $M_1 \in \mathcal{K}$ such that $M_1 \prec\!\!\prec M$, so that the conditions of belonging of L_M or K_M with any $M \in \mathcal{K}$ are equivalent.

Example 1.3. $M_\alpha(s) = \exp(s/\ln^\alpha s)$, $M_\alpha \in \mathcal{K}$ at $\alpha \leqslant 1$. The corresponding function $\omega(p) = \mathbf{Sc}_\infty[M_\alpha](p) \overset{\mathcal{L}}{\sim} p\ln^\alpha p$. Thus, if we formulate the condition on g as $\|g\|_{L_p(Q_T)} \leqslant Cp\ln^\alpha p$, the statement of the type of the Gronwall lemma for (9) is valid for $\alpha \leqslant 1$. One can continue to refine the conditions on M by choosing, for example, $M(s) = \exp(s/(\ln s \ln^\alpha \ln s))$, which will give new logarithms in ω (compare [2]), etc.

The properties of some differential operators in the Orlicz spaces are also needed. Since, for L_p, these properties are well known, it is natural to employ extrapolation methods. If we use the representation of the Orlicz spaces in the form of the spaces $L_{\omega,\infty}$, there is no need for these methods because of the triviality of the problem but this representation is cumbersome and, hence, inconvenient. For the purposes of the present work, it is convenient to use the constructive extrapolation method developed in [13–15], which consists of the following. Let a linear operator A act boundedly in L_p for all $p \gg 1$ and its norm $\|A\|_{\mathcal{L}(L_p)} \leqslant C\varphi(p)$. Then, one should calculate the inverse Mellin transformation on the function $\varphi^p(p)$:

$$\varphi^p(p) = \int_{\sigma}^{+\infty} \psi(s) s^p \, ds \tag{1.4}$$

($\sigma \geqslant 0$ can be chosen arbitrarily; moreover, formula (1.4) is extended in some sense to the case of nonanalytical functions φ [15]) and use the obtained function ψ as the kernel of the convolution type integral transformation:

$$\mathbf{F}_{\psi,\sigma}[\Phi](v) = \int_{\sigma}^{+\infty} \psi(s)\Phi(vs)\, ds. \tag{1.5}$$

As a result, it can be argued that A acts boundedly from L_M to L_Φ for any N-functions Φ and M linked by the relation $M = \mathbf{F}_{\psi,\sigma}[\Phi]$. In particular, if $\varphi(p) = p$, then, relation (1.4) (to within the relation $\overset{\mathcal{L}}{\sim}$, which is insignificant as noted above) implies that $\psi(s) = \mathrm{e}^{-s}$, and the operator (1.5) becomes the operator $\mathbf{S}$, which was studied in [13]:

$$\mathbf{S}[\Phi](v) = \int_{0}^{+\infty} \mathrm{e}^{-s}\, \Phi(vs)\, ds. \tag{1.6}$$

Thus, we obtain the following statement.

Statement 1.2. *If the linear operator A for all $p \gg 1$ has the property $\|A\|_{\mathcal{L}(L_p)} \leqslant Cp$, then $A \in \mathcal{L}(L_M, L_\Phi)$ for all N-functions M and Φ linked by the relation $M = \mathbf{S}[\Phi]$, where the operator $\mathbf{S}$ is defined by formula* (1.6).

In the present work, Statement 1.2 is considered in the case $\Phi \in \mathcal{K}$.

Statement 1.3. *If $M = \mathbf{S}[\Phi]$ and M satisfies the Δ^2-condition, then $\Phi \in \mathcal{K}$ is equivalent to $M \in \mathcal{K}_1$ where $\mathcal{K}_1$ consists of functions M that satisfy the condition*

$$\int^{+\infty} \frac{\ln\ln M(s)}{s^2}\, ds = +\infty. \tag{1.7}$$

Remark 1.2. As follows from Remark 1.1 and the fact that $\mathbf{S}$ increases the rate of increase of the functions to which it is applied, the Δ^2-condition in Statement 1.3 is not an additional strong constraint but it only eliminates the "pathological" functions insignificant for applications.

Proof of Statement 1.3. We first construct a convenient representation for the asymptotics of the function M. For this, we consider the equation $\Phi(vs_*(v)) = \exp(s_*(v)/2)$ for the quantity $s_*(v)$ as $v \gg 1$. Since, for $s = 1$, the quantity $\mathrm{e}^{-s/2}\,\Phi(vs) > 1$, and for $s \to +\infty$ it tends to zero (since the operator $\mathbf{S}$ is defined on functions growing more slowly than polynomial), such a quantity $s_*(v)$ exists. It is also easy to show that it is unique for $v \gg 1$ and tends monotonically to $+\infty$ as $v \to +\infty$. Thus, for $s > s_*(v)$, we have $\Phi(vs) < \mathrm{e}^{s/2}$, so that

$$M(v) = \int_{0}^{s_*(v)} \mathrm{e}^{-s}\, \Phi(vs)\, ds + \int_{s_*(v)}^{+\infty} \mathrm{e}^{-s}\, \Phi(vs)\, ds \leqslant \Phi(vs_*(v)) + 2\exp(-s_*(v)/2) \leqslant 2\exp(s_*(v)/2).$$

Thus, $M(v) \prec \exp(s_*(v))$. At the same time, if we set

$$\beta(v) = \frac{1}{4v}\Phi^{-1}\Big(\frac{1}{2}\,\Phi(vs_*(v))\Big),$$

by the definition, we obtain $\beta(v) \leqslant s_*(v)/4$ and

$$M(4v) \geqslant \int_{\beta(v)}^{+\infty} \mathrm{e}^{-s}\, \Phi(4vs)\, ds \geqslant \Phi(4v\beta(v))\, \mathrm{e}^{-\beta(v)} = \frac{1}{2}\, \exp\Big(\frac{s_*(v)}{2} - \beta(v)\Big) \geqslant \frac{1}{2}\exp(s_*(v)/4),$$

whence, by virtue of the Δ^2-condition for M, it follows that $M(v) \succ \exp(s_*(v))$.

Thus, $M(v) \sim \exp(s_*(v))$. We can now write the following sequence of relations, in which the symbol "$\overset{+\infty}{\sim}$" denotes the simultaneous tension to $+\infty$ (or the assumption of finite values):

$$\int^{+\infty} \frac{\ln \Phi(s)}{s^2}\, ds = \int^{+\infty} \frac{\ln \Phi(vs_*(v))}{v^2 s_*^2(v)}\, d(vs_*(v)) = \int^{+\infty} \frac{1}{2v}\, \frac{d(vs_*(v))}{vs_*(v)}$$

$$= \frac{1}{2v}\, \ln\,(vs_*(v))\Big|_{v=+\infty} + \frac{1}{2} \int^{+\infty} \frac{\ln(vs_*(v))}{v^2}\, dv \overset{+\infty}{\sim} \int^{+\infty} \frac{\ln v + \ln s_*(v)}{v^2}\, dv$$

$$\overset{+\infty}{\sim} \int^{+\infty} \frac{\ln s_*(v)}{v^2}\, dv = \int^{+\infty} \frac{\ln \ln M(v)}{v^2}\, dv,$$

which was to be proved.

2. Existence and Uniqueness of a Generalized Solution with Unbounded Vorticity. As is noted in [2] (without proof), the definition of the generalized solution of problem (1)–(4) given in [5] and the existence theorem for it can be extended to the case of unbounded vorticity. Therefore, in the present work, we do not give a detailed proof of the existence and only note the differences arising in the case considered and give the necessary data from [5].

As in [5], the problem should be reduced to finding the stream function ψ:

$$\boldsymbol{v} = \hat{\nabla}\psi \equiv \Big(\frac{\partial \psi}{\partial x_2}, -\frac{\partial \psi}{\partial x_1}\Big).$$

In this case, by virtue of (4), it can be assumed that $\psi\Big|_{\partial\Omega} = 0$. Applying the rot operation [which is understood here as a scalar operator: $\operatorname{rot} \boldsymbol{w} = (w_2)_{x_1} - (w_1)_{x_2} = -\hat{\nabla} \cdot \boldsymbol{w}$] to (1) and taking into account (2), we obtain the Helmholtz equation for the vorticity $\omega = \operatorname{rot} \boldsymbol{v} = -\Delta\psi$:

$$\frac{\partial \omega}{\partial t} + \boldsymbol{v} \cdot \nabla\omega = \operatorname{rot} \boldsymbol{f}.$$

Thus, we obtain the following problem for the stream function:

$$\frac{\partial\, \Delta\psi}{\partial t} + \hat{\nabla}\psi \cdot \nabla\Delta\psi = -\operatorname{rot} \boldsymbol{f}, \qquad \psi\Big|_{t=0} = \psi_0, \quad \psi\Big|_{\partial\Omega} = 0, \tag{2.1}$$

where ψ_0 is the stream function of the initial velocity $\boldsymbol{v}_0$.

We fix an arbitrary function $M \in \mathcal{K}_1$ and introduce the following functional spaces.

Definition 2.1. $V^{(M)}$ is the space of generalized solutions u of problems of the form

$$\Delta u = g, \qquad u\Big|_{\partial\Omega} = 0,$$

where the functions g are any elements of the space $L_M(\Omega)$. In this case, $\|u\|_{V^{(M)}} = \|\Delta u\|_{L_M(\Omega)}$.

Definition 2.2. $V_1^{(M)}$ is the space of functions which are determined in Q_T, belong to $V^{(M)}$ for almost all $t \in (0, T)$, and have a finite norm $\|u\|_{V_1^{(M)}} = \sup\limits_{t\in(0,T)} \|u(t, \cdot)\|_{V^{(M)}}$.

In the limiting case where M tends to infinity for finite values of the argument [i.e., $L_M(\Omega) = L_\infty(\Omega)$], these spaces coincide with the spaces V and V_1 introduced in [5]. By analogy with [5], we give the following definition.

Definition 2.3. Let $\psi_0 \in V^{(M)}$ and $\operatorname{rot} \boldsymbol{f} \in L_1(0, T, L_M(\Omega))$. The generalized solution of problem (2.1) is a function $\psi \in V_1^{(M)}$ that satisfies the integral identity

$$\int_\Omega \chi\Big|_{t=0} \Delta\psi_0\, d\boldsymbol{x} + \int_{Q_T} \Delta\psi \Big[\frac{\partial \chi}{\partial t} + \hat{\nabla}\psi \cdot \nabla\chi\Big]\, d\boldsymbol{x}\, dt = \int_{Q_T} \chi \operatorname{rot} \boldsymbol{f}\, d\boldsymbol{x}\, dt$$

for all functions χ which are smooth in Q_T and vanish on the lateral surface $\partial\Omega \times (0, T)$ and for $t = T$.

It can be shown, as in [5], that, in the sense of definition 2.3, the generalized solution ψ of problem (2.1) is weakly continuous in t in the space $W_2^1(\Omega)$ [and in this sense, in particular, it satisfies the initial condition in (2.1)] and has derivatives $\nabla\psi_t \in L_\infty(0, T, L_p(\Omega))$ for all $p < \infty$ (although, for these derivatives, the last term in the estimate (3.14) of [5] changes).

The operator $A:\ \Delta\psi\mapsto D^2_{\boldsymbol{x}}\psi$ is bounded in all $L_p(\Omega)$ and its norm $\|A\|_{\mathcal{L}(L_p)}\leqslant Cp$ (the notation $D^2_{\boldsymbol{x}}h=\nabla_{\boldsymbol{x}}\otimes\nabla_{\boldsymbol{x}}h$ denotes the second derivatives of h with respect to $\boldsymbol{x}$). In [5], this fact was used for the simple extrapolation derivation: $A\in\mathcal{L}(L_\infty(\Omega),L_N(\Omega))$, where $N(s)=\mathrm{e}^s$, and $L_N(\Omega)$ was represented as $L_{\omega,\infty}$ with $\omega(p)=p$. In the case considered, one needs the more general statement implied by statement 1.2: $A\in\mathcal{L}(L_M(\Omega),L_\Phi(\Omega))$, where $M=\mathbf{S}[\Phi]$. According to statement 1.3, the assumption $M\in\mathcal{K}_1$ is equivalent to the fact that $\Phi\in\mathcal{K}$. Thus, we proved the following statement.

Statement 2.1. *For the generalized solution of problem* (2.1), *it is true that* $D^2_{\boldsymbol{x}}\psi\in L_\infty(0,T,L_\Phi(\Omega))$, *where* $\Phi\in\mathcal{K}$.

The further proof of the existence of the generalized solution almost literally follows the reasoning in [5], and this proof implies an existence theorem similar to theorem 4.1 in [5], with the only difference that one should replace V by $V^{(M)}$ and require that $\operatorname{rot}\boldsymbol{f}$ belong only to the space $L_1(0,T,L_M(\Omega))$. Indeed, the desired solution is constructed as the fixed point of the operator B: $\psi\mapsto\psi'$ of the solution to the problem

$$\frac{\partial\Delta\psi'}{\partial t}+\hat{\nabla}\psi\cdot\nabla\Delta\psi'=-\operatorname{rot}\boldsymbol{f},\qquad \psi'\Big|_{t=0}=\psi_0,\quad \psi'\Big|_{\partial\Omega}=0. \tag{2.2}$$

As shown in [5], for appropriately determined solutions of this problem, the estimate

$$\max_{t\in[0,T]}\|\Delta\psi'(t,\cdot\,)\|_{L_p}\leqslant\|\Delta\psi_0\|_{L_p}+\int\limits_0^T\|\operatorname{rot}\boldsymbol{f}(t,\cdot\,)\|_{L_p}\,dt \tag{2.3}$$

holds for all $p=2k$, $k\in\mathbb{N}$. Dividing (2.3) by $\omega(p)$ and transforming to $\sup_p$, we obtain the estimate

$$\|\Delta\psi'\|_{L_\infty(0,T,L_{\omega,\infty})}\leqslant\|\Delta\psi_0\|_{L_{\omega,\infty}}+\|\operatorname{rot}\boldsymbol{f}\|_{L_1(0,T,L_{\omega,\infty})}.$$

Using the equivalent representation $L_{\omega,\infty}=L_M$, we obtain the estimate

$$\|\psi'\|_{V_1^{(M)}}\leqslant\|\psi_0\|_{V^{(M)}}+\|\operatorname{rot}\boldsymbol{f}\|_{L_1(0,T,L_M(\Omega))},$$

which is an analog of estimate (4.17) in [5]. The further reasoning about the fixed point of the operator B of problem (2.2) are repeated literally.

The main purpose of Sec. 2 is to prove the uniqueness theorem.

Theorem 2.1. *For* $M\in\mathcal{K}_1$ [*i.e., if* (1.7) *is satisfied*], *the generalized solution of problem* (2.1) *in the class* $V_1^{(M)}$ (*i.e., in the sense of definition* 2.3) *is unique.*

Proof. As was shown in [5], for the difference $\alpha=\psi_1-\psi_2$ of two solutions, the following inequality holds:

$$\int\limits_\Omega|\nabla\alpha|^2\,d\boldsymbol{x}\leqslant\int\limits_0^t\int\limits_\Omega|D^2_{\boldsymbol{x}}\psi_1|\cdot|\nabla\alpha|^2\,d\boldsymbol{x}\,ds. \tag{2.4}$$

By virtue of statement 2.1, $g=|D^2_{\boldsymbol{x}}\psi_1|\in L_\infty(0,T,L_\Phi(\Omega))$ with $\Phi\in\mathcal{K}$. Replacing, if necessary, Φ by some other $\Phi_1\in\mathcal{K}$, we achieve the inclusion $g\in K_{\Phi_1}(Q_T)$. In addition, by virtue of statement 2.1, $|\nabla\alpha|$ is bounded. In view of Statement 1.1, from (2.4) we obtain $\nabla\alpha=0$, and, hence, $\alpha=0$. Theorem 2.1 is proved.

The further reasoning about the determination of the pressure p and the construction of the unique generalized solution of the initial problem (1)–(4) follow the logic of [5]. Thus, we proved the following theorem.

Theorem 2.2. *The solution of problem* (1)–(4) *in the class* $\operatorname{rot}\boldsymbol{v}\in L_\infty(0,T,L_M(\Omega))$, $M\in\mathcal{K}_1$ [*see* (1.7)] *exists and is unique* (*the pressure* p *is determined to within an additive function of time*).

As noted in [2], for the proof of the uniqueness of solutions to (1)–(4), the dimension of the flow is insignificant. We explain how theorem 2.2 is extended to the case $n=3$. If problem (1)–(4) has two solutions $(\boldsymbol{v}_1,p_1)$ and $(\boldsymbol{v}_2,p_2)$, for their difference $(\boldsymbol{v},p)$ we have the problem

$$\frac{\partial\boldsymbol{v}}{\partial t}+(\boldsymbol{v}_1\cdot\nabla)\boldsymbol{v}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}_2+\nabla p=0,\qquad \operatorname{div}\boldsymbol{v}=0; \tag{2.5}$$

$$\boldsymbol{v}\Big|_{t=0}=0,\qquad \boldsymbol{v}\cdot\boldsymbol{n}\Big|_{\partial\Omega\times(0,T)}=0. \tag{2.6}$$

Multiplying the first equation in (2.5) scalarly by $2\boldsymbol{v}$ and integrating over $Q_t=\Omega\times(0,t)$, in view of the second equation in (2.5) and conditions (2.6), we obtain

$$\int\limits_{\Omega} |\boldsymbol{v}|^2 \, d\boldsymbol{x} = -2 \int\limits_0^t \int\limits_{\Omega} (\boldsymbol{v} \otimes \boldsymbol{v}) : \boldsymbol{D}(\boldsymbol{v}_2) \, d\boldsymbol{x} \, ds, \tag{2.7}$$

where $\boldsymbol{D}$ is the symmetric part of the gradient (strain rate tensor). Equality (2.7) is obtained formally but, for the solutions of the class considered, this procedure can be substantiated rigorously [2, 5]. Thus, (2.7) implies an inequality of the form (9) with $\psi = |\boldsymbol{v}|^2$ and $g = C|\boldsymbol{D}(\boldsymbol{v}_2)|$. By analogy with the two-dimensional case, we formulate the constraint on the solution in terms of the vorticity $\boldsymbol{\omega} = \operatorname{rot} \boldsymbol{v}$. The operator of the problem for finding $\boldsymbol{v}$ for prescribed $\boldsymbol{\omega}$:

$$\operatorname{rot} \boldsymbol{v} = \boldsymbol{\omega}, \qquad \operatorname{div} \boldsymbol{v} = 0, \qquad \boldsymbol{v} \cdot \boldsymbol{n}\Big|_{\partial\Omega} = 0, \tag{2.8}$$

or, more precisely, the operator $\boldsymbol{\omega} \mapsto \nabla \otimes \boldsymbol{v}$ [in (2.7), an estimate for $\boldsymbol{D}(\boldsymbol{v})$ is required, but, in this case, it is not better than that for $\nabla \otimes \boldsymbol{v}$] for $n = 3$ has the same properties (used above) as for $n = 2$: its norm in $\mathcal{L}(L_p)$ does not exceed Cp [16]. Consequently, to satisfy the equality $\boldsymbol{v} = 0$, it is again necessary to require that $\boldsymbol{D}(\boldsymbol{v}_2) \in K_\Phi(Q_T)$, $\Phi \in \mathcal{K}$, which leads to the requirement $\boldsymbol{\omega} \in L_\infty(0, T, L_M(\Omega))$, $M \in \mathcal{K}_1$ [i.e., (1.7)]. Thus, theorem 2.2 is also true for $n = 3$.

It is natural to compare the result of theorem 2.2 with the result of [2]. If the result of [2] is represented in the notation used in the present work, it will be formulated as follows. We consider solutions of the class $\operatorname{rot} \boldsymbol{v} \in L_\infty(0, T, L_{\theta,\infty})$ (here, $\overline{\lim}$ is used instead of sup in (1.1), which, in this case, is insignificant) in which θ is subject to the constraint

$$\int^{+\infty} \frac{da}{a\rho(a)} = \infty, \tag{2.9}$$

where $\rho(a) = \inf\limits_{\varepsilon \in [0, \varepsilon_0]} a^\varepsilon \theta(1/\varepsilon)/\varepsilon$ for $a \gg 1$. As is indicated in Sec. 1, this condition on $\boldsymbol{v}$ is equivalent to the constraint $\operatorname{rot} \boldsymbol{v} \in L_\infty(0, T, L_M(\Omega))$, where $M(v) = \int\limits_1^{+\infty} \frac{v^p \, dp}{\theta^p(p)}$, i.e., $M = \mathbf{In}_\infty[\theta]$. Thus, the problem of the relation between conditions (1.7) and (2.9) arises. We give a more convenient description of the class of admissible θ, i.e., those which satisfy (2.9).

Lemma 2.1. *Condition* (2.9) *is equivalent to the divergence of the integral*

$$\int^{+\infty} \frac{d\xi}{\xi\theta(\xi)} = +\infty. \tag{2.10}$$

Proof. We have $\rho(a) = \inf\limits_{z \gg 1} a^{1/z} z\theta(z) = a^{1/z_*} z_* \theta(z_*)$, where $1 + \mathbf{e}_\theta(z_*) = (\ln a)/z_*$; $\mathbf{e}_\theta(z) = z\theta'(z)/\theta(z)$ is an exponential characteristic of the function θ (see [14, 15]). As is noted in [2], condition (2.9) obviously implies that, at infinity, the function θ increases more slowly than $\ln^{1+\varepsilon}$ [it is easy to see that the same statement is also true for (2.10)], i.e., $\mathbf{e}_\theta(z)$ decreases as $1/\ln z$, so that z_* has the asymptotic representation $z_* = (\ln a)(1 - o(1))$. The equivalence of (2.9) and (2.10) is now obvious. Lemma 2.1 is proved.

Example 2.1. If $\theta(p) = \ln^\alpha p$, $\alpha > 0$, the condition (2.10) is satisfied for $\alpha \leqslant 1$. In addition, the class (2.9) [or what is the same, the class (2.10)] contains functions of the form $\theta(p) = \ln p \ln \ln p \ldots$ (see [2]).

Example 2.2. We consider the function $M(v) = \exp(\exp(v^\gamma))$, $\gamma > 0$. Clearly, (1.7) is satisfied for $\gamma \geqslant 1$. It is easy to show that $\theta(p) = \mathbf{Sc}_\infty[M](p) \overset{\mathcal{L}}{\sim} \ln^{1/\gamma} p$. Thus, condition (2.9) is also satisfied for $\gamma \geqslant 1$.

The following lemma can be formulated.

Lemma 2.2. *Condition* (1.7) *is equivalent to the divergence of the integral*

$$\int^{+\infty} \frac{ds}{sM^{-1}(\mathrm{e}^s)} = +\infty. \tag{2.11}$$

Proof. As is proved in [9] [see also (1.3)], the following conditions are equivalent:

$$\int^{+\infty} \frac{\ln N(s)}{s^2} = +\infty \qquad \Longleftrightarrow \qquad \int^{+\infty} \frac{ds}{sN^{-1}(s)} = +\infty.$$

Changing $N(s) = \ln M(s)$, we obtain the required result.

Preliminary construction is completed by the following asymptotic representation.

Lemma 2.3. *For the functions θ of the class* (2.9) [*or what is the same, the class* (2.10)] *and the functions M of the class* (1.7) [*or, what is the same, the class* (2.11)] *linked by the relation $M = \mathbf{In}_\infty[\theta]$ (i.e., $\theta = \mathbf{Sc}_\infty[M]$), the asymptotics $\theta(s) \overset{\mathcal{L}}{\sim} M^{-1}(\mathrm{e}^s)$ holds.*

Proof. We introduce the following notation: $N = M^{-1}$; $\varkappa$ is a function that describes the solution of the equation $\mathbf{e}_N(\varkappa(p)) = 1/p$ (as shown in [14], for the functions of the class considered, this equation has a unique solution; therefore, the function $\varkappa$ is correctly determined). Then, $\theta(p) = N(\varkappa(p))\varkappa^{-1/p}(p)$ because $\theta(p) = \max M^{-1}(v)/v^{1/p}$. For the function $M(u) = \exp(\exp(u^\gamma))$ from example 2.2, it is easy to calculate $\ln \varkappa(p) \sim (p/\gamma)/\ln(p/\gamma)$, so that, in the class considered, $\ln \varkappa(p)$ obviously increases more slowly than p. From this, in particular, it follows that $\ln \varkappa(p) - p < 0$ at $p \gg 1$. We examine the quantity

$$A \equiv \ln N(\varkappa(p)) - \ln N(\mathrm{e}^p) = \int\limits_{\mathrm{e}^p}^{\varkappa(p)} \frac{N'(\xi)\, d\xi}{N(\xi)} = \int\limits_{\varkappa^{-1}(\mathrm{e}^p)}^{p} \frac{N'(\varkappa(\eta))\varkappa'(\eta)\, d\eta}{N(\varkappa(\eta))} = \int\limits_{\varkappa^{-1}(\mathrm{e}^p)}^{p} \frac{\varkappa'(\eta)\, d\eta}{\eta\varkappa(\eta)}.$$

Estimating the multiplier $\eta \in [\varkappa^{-1}(\mathrm{e}^p), p]$ in the denominator and calculating the integral, we obtain the estimate

$$A \in \Big[\frac{\ln \varkappa(p)}{p} - 1, \frac{\ln \varkappa(p) - p}{\varkappa^{-1}(\mathrm{e}^p)}\Big].$$

Thus,

$$-1 \leqslant \ln N(\varkappa(p)) - \ln N(\mathrm{e}^p) - \frac{\ln \varkappa(p)}{p} \leqslant \frac{\ln \varkappa(p) - p}{\varkappa^{-1}(\mathrm{e}^p)} - \frac{\ln \varkappa(p)}{p} < 0$$

(we note that the last expression tends to zero as $p \to \infty$), whence we finally obtain $\mathrm{e}^{-1} \leqslant \theta(p)/N(\mathrm{e}^p) \leqslant 1$, which was to be proved.

Using Lemmas 2.1–2.3, we finally obtain the following statement.

Statement 2.2. *Conditions* (2.9) *and* (1.7) *are equivalent.*

Thus, the result of Theorem 2.2 coincides with result of [2]. The equivalence of these results agrees with the unimprovability of the conditions on the function g in inequality (9), which is proved in different terms in [2] (in terms reduced to the spaces $L_{\theta,\infty}$) and [9] (in terms of Orlicz spaces). However, in our opinion, an advantage of the result obtained in the present work is that it is clearer and more convenient for the verification of the conditions of the theorem.

3. Existence of Weakly Regular Solutions of the Plane Problem (1)–(4). We consider problem (1)–(4) for $n = 2$ and find conditions on the input data, especially on the initial data $\boldsymbol{v}_0$, for which it is possible to prove the existence of solutions. The class of solutions obtained in this case is much wider than that considered in Sec. 2 and the uniqueness of these solutions would be difficult to prove.

As noted in the introduction, the problem in question was studied in [8], where the class (6), (7) was formulated. Some designations used in the following definition were taken from [8].

Definition 3.1. $S(\Omega)$ is a set of smooth vector fields $\boldsymbol{v}$ specified in Ω and satisfying (2) and (4); $S_2(\Omega)$ is the closure of $S(\Omega)$ in the norm $L_2(\Omega)$.

The solutions of the class considered in Sec. 3 are described in the following definition.

Definition 3.2. Let $\boldsymbol{v}_0 \in S_2(\Omega)$ and $\boldsymbol{f} \in L_2(Q_T)$. A function $\boldsymbol{v} \in L_2(0, T, S_2(\Omega))$ is called a weak generalized solution of problem (1)–(4) if it satisfies the identity

$$\int\limits_{Q_T} \boldsymbol{v} \cdot \Big[\frac{\partial \boldsymbol{\xi}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{\xi}\Big]\, d\boldsymbol{x}\, dt + \int\limits_{Q_T} \boldsymbol{f} \cdot \boldsymbol{\xi}\, d\boldsymbol{x}\, dt + \int\limits_{\Omega} \boldsymbol{v}_0 \cdot \boldsymbol{\xi}\Big|_{t=0}\, d\boldsymbol{x} = 0 \tag{3.1}$$

for all $\boldsymbol{\xi} \in C^\infty(Q_T)$ such that $\boldsymbol{\xi}\Big|_{t=T} = 0$, $\boldsymbol{\xi}(t, \cdot\,) \in S(\Omega)$ for all $t \in [0, T]$.

Definition 3.2 does not contain p but the pressure is uniquely (to within an additive function of t) retrieved in $\boldsymbol{v}$ on the basis of the orthogonal decomposition $L_2(\Omega) = S_2(\Omega) \oplus G(\Omega)$ [17] which also makes definition 3.2 correct.

The main purpose of Sec. 3 is to prove the following theorem.

Theorem 3.1. *Let an N-function M be such that*

$$\bar{M}(s) \prec\!\!\prec \exp(s^2). \tag{3.2}$$

Then, for any input data of the class

$$\operatorname{rot} \boldsymbol{v}_0 \in L_M(\Omega); \tag{3.3}$$

$$\operatorname{rot} \boldsymbol{f} \in L_1(0,T,L_M(\Omega))$$

a generalized weak solution of problem (1)–(4) *in the sense of definitions* 3.2 *exists and*

$$\boldsymbol{v} \in L_\infty(0,T,S_2(\Omega)), \qquad \operatorname{rot} \boldsymbol{v} \in L_\infty(0,T,L_M(\Omega)); \tag{3.4}$$

and if M satisfies the Δ_2-condition, the following estimate holds:

$$\|\operatorname{rot} \boldsymbol{v}\|_{L_\infty(0,T,L_M(\Omega))} \leqslant \|\operatorname{rot} \boldsymbol{v}_0\|_{L_M(\Omega)} + \|\operatorname{rot} \boldsymbol{f}\|_{L_1(0,T,L_M(\Omega))}. \tag{3.5}$$

REMARK 3.1. By virtue of the non-faster-than-polynomial growth of the functions M considered, the Δ_2-condition has a purely technical nature and does not impose significant constraints on M.

Proof of Theorem 3.1. We denote $Y = \{\, \boldsymbol{u} \in S_2(\Omega)\colon \operatorname{rot} \boldsymbol{u} \in L_M(\Omega)\}$, $\|\boldsymbol{u}\|_Y = \|\boldsymbol{u}\|_{L_2(\Omega)} + \|\operatorname{rot} \boldsymbol{u}\|_{L_M(\Omega)}$. According to the condition, $\boldsymbol{v}_0 \in Y$. We consider a sequence of functions $\boldsymbol{v}_{0k} \in S(\Omega)$ such that $\boldsymbol{v}_{0k} \to \boldsymbol{v}_0$ weak-star in Y. For example, as the functions $\boldsymbol{v}_{0k}$, we may use averagings of $\boldsymbol{v}_0$. For each k, it is possible to construct a unique generalized solution $\boldsymbol{v}_k$ of problem (1)–(4) (with $\boldsymbol{v}_{0k}$ instead of $\boldsymbol{v}_0$) in the sense of the definition in Sec. 2; in this case, the following estimate holds:

$$\|\operatorname{rot} \boldsymbol{v}_k\|_{L_\infty(0,T,L_M(\Omega))} \leqslant \|\operatorname{rot} \boldsymbol{v}_{0k}\|_{L_M(\Omega)} + \|\operatorname{rot} \boldsymbol{f}\|_{L_1(0,T,L_M(\Omega))} \tag{3.6}$$

(for this estimate see, for example, [8]), and $\boldsymbol{v}_k$ and $\boldsymbol{v}_{0k}$ satisfy identity (3.1). These approximate solutions are smooth enough (with a large margin) for the first energy estimate.

The key role in the present construction is played by the compact embedding $Y \hookrightarrow\hookrightarrow L_2(\Omega)$, which follows from the following consideration. As noted in [18], $\mathring{W}_2^1(\Omega) \hookrightarrow\hookrightarrow L_{\bar{M}}(\Omega)$ if condition (3.2) is satisfied; therefore, $L_M(\Omega) \hookrightarrow\hookrightarrow W_2^{-1}(\Omega)$. By virtue of the method of compensated compactness (see [19]), we obtain the required embedding $Y \hookrightarrow\hookrightarrow L_2(\Omega)$. By virtue of (3.6) and the first energy estimate, the set $\{\boldsymbol{v}_k\}$ is bounded in the space $L_\infty(0,T,Y)$; therefore, there exists $\boldsymbol{v} \in L_\infty(0,T,Y)$ such that $\boldsymbol{v}_k \to \boldsymbol{v}$ weak-star in this space (after selection of the subsequence). Thus, this function satisfies (3.4). If M satisfies the Δ_2-condition, it is possible to achieve that $\operatorname{rot} \boldsymbol{v}_{0k} \to \operatorname{rot} \boldsymbol{v}_0$ in the norm $L_M(\Omega)$; then, in (3.6), we can pass to the limit and obtain (3.5).

Identity (3.1) for $\boldsymbol{v}_k$ implies that $\partial \boldsymbol{v}_k/\partial t$ act on solenoidal finite functions in the same way as the expression $\boldsymbol{f} - \operatorname{div}(\boldsymbol{v}_k \otimes \boldsymbol{v}_k)$, i.e., these quantities coincide as functionals above the space of the corresponding test functions. Thus, $\{\partial \boldsymbol{v}_k/\partial t\}$ is bounded in $L_2(0,T,X^*)$, where $X = \{\boldsymbol{\xi} \in \mathring{W}_\infty^1(\Omega),\ \operatorname{div} \boldsymbol{\xi} = 0\}$ [X^* is a special extension of the space $W_1^{-1}(\Omega)$]. By virtue of the embeddings $Y \hookrightarrow\hookrightarrow L_2(\Omega) \hookrightarrow\hookrightarrow X^*$ and the Aubin–Simon considerations of compactness [20, 21], it can be argued that $\{\boldsymbol{v}_k\}$ is compact in $L_2(Q_T)$. Thus, it can be assumed that $\boldsymbol{v}_k \to \boldsymbol{v}$ strongly in $L_2(Q_T)$, but, in this case, the passage to the limit in identity (3.1) for $\boldsymbol{v}_k$ is trivial, and, consequently, $\boldsymbol{v}$ is the desired solution. The theorem is proved.

REMARK 3.2. From the boundedness of $\{\partial \boldsymbol{v}_k/\partial t\}$ in $L_2(0,T,X^*)$ and $\{\boldsymbol{v}_k\}$ in $L_\infty(0,T,Y)$, using a standard reasoning, we obtain the convergence $\boldsymbol{v}_k(t) \to \boldsymbol{v}(t)$ in the norm $L_2(\Omega)$ for almost all $t \in [0,T]$. Therefore, in definition 3.2, one need not require the condition $\boldsymbol{\xi}\Big|_{t=T} = 0$ and can take the integral over Q_t instead of Q_T. This gives rise to the additional integral $\int\limits_\Omega \boldsymbol{v} \cdot \boldsymbol{\xi}\, d\boldsymbol{x}$, in which passage to the limit is also possible (this type of definition of the solution was proposed in [2]).

REMARK 3.3. From the proof of Theorem 3.1, it follows that the class of flows considered in it is extremely wide and still admissible for the property $\boldsymbol{v}(t,\cdot) \in L_2(\Omega)$, which implies that the flow kinetic energy (over the entire domain and subdomains) is finite if the class is formulated in terms of vorticity.

The problem arises of comparing the result of Theorem 3.1 with the result of [8], i.e., the problem of the relation between conditions (7) and (3.2). This comparison is especially important since different methods were used. In [8], the compactness estimate was derived by analyzing the properties of solutions to problem (2.8) using the singular integral:

$$\boldsymbol{v}(\boldsymbol{x}) = \int\limits_\Omega \boldsymbol{g}(\boldsymbol{x},\boldsymbol{y})\omega(\boldsymbol{y})\, d\boldsymbol{y}. \tag{3.7}$$

In this case, according to Lemma 1 in [8], the constant γ in (7) is calculated as follows: $\gamma = 4A$, where A is the constant in the estimate mes $\{k(\boldsymbol{z}) > t\} \leqslant (A/t)^2$, and $k(\boldsymbol{z}) \leqslant C/|\boldsymbol{z}|$, $|\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y})| \leqslant C/|\boldsymbol{x} - \boldsymbol{y}|$. Thus, $\gamma = 4C\sqrt{\pi}$, where C is the constant in the estimate of the kernel $\boldsymbol{g}$ in (3.7), which depends on the diameter of Ω. For the class considered in the present paper, we can write $\bar{M}(s) = \exp(s^2/\nu(s))$, where ν is a function that, at infinity, increases more slowly than the quadratic function. For simplicity, we confine ourselves to monotonically increasing functions ν; then, the limit $\lim\limits_{s\to+\infty} \nu(s) \leqslant +\infty$ exists. It is easy to show that condition (3.2) is equivalent to the requirement

$$\lim_{s\to+\infty} \nu(s) = +\infty. \tag{3.8}$$

At the same time, condition (7) is written as

$$\int^{+\infty} \exp\Big(\frac{t^2}{\nu(t)} - \frac{t^2}{\gamma}\Big) \frac{2 - \mathbf{e}_\nu(t)}{t\nu(t)}\, dt < +\infty.$$

By virtue of the polynomial nature of the asymptotics of the fraction $(2 - \mathbf{e}_\nu(t))/(t\nu(t))$ at infinity, this condition implies that

$$\lim_{s\to+\infty} \nu(s) > \gamma. \tag{3.9}$$

Thus, for any fixed domain Ω, condition (7) [or, what is the same, condition (3.9)] is weaker (more general) than condition (3.2) [or what is the same, condition (3.8)], but if the result is considered in the class of all domains Ω simultaneously, these conditions are equivalent.

Advantages of Theorem 3.1 are the clearer form of condition (3.2) and the weaker requirements for $\boldsymbol{f}$ than those in [8]. In addition, in Theorem 3.1 (in contrast to [8]), the Δ_2-condition for M, generally speaking, is not required, if there is no need to obtain estimate (3.5).

In conclusion, we give a partial physical interpretation of the results obtained. We construct a solenoidal field $\boldsymbol{u}$ in $\mathbb{R}^2$ which belongs to $L_2(\Omega)$ and has vorticity with a singularity of specified nature in the vicinity of zero (for definiteness, we assume that $0 \in \Omega$). We will seek this field in the form $\boldsymbol{u} = \hat{\nabla}\psi$, where $\psi = \psi(r)$, $r = |\boldsymbol{x}|$. Then,

$$\omega = \omega(r) = \operatorname{rot} \boldsymbol{u} = -\Delta\psi(r) = -\psi''(r) - \psi'(r)/r,$$

where the prime denotes differentiation with respect to r. If ω is specified, the stream function ψ is expressed as

$$\psi'(r) = \frac{A}{r} - \frac{1}{r}\int_0^r \xi\omega(\xi)\, d\xi$$

with an arbitrary constant A. Obviously,

$$\boldsymbol{u} = \psi'(r) \cdot \begin{bmatrix} x_2/r \\ -x_1/r \end{bmatrix},$$

i.e., $|\boldsymbol{u}| = \psi'(r)$. Since we consider flows with integrable vorticity, it follows that $\int\limits_0^r \xi\omega(\xi)\, d\xi \to 0$ as $r \to 0$; therefore, $\psi'(r) = (A + o(1))/r$, and for the embedding $\boldsymbol{u} \in L_2(\Omega)$, it is necessary and sufficient that $A = 0$. Thus, for any specified function $\omega = \omega(r)$, it is possible to construct a field

$$\boldsymbol{u} = \frac{1}{r^2}\int_0^r \xi\omega(\xi)\, d\xi \cdot \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} \tag{3.10}$$

with vorticity ω and finite energy. The condition $\omega \in L_M(\Omega)$ implies the convergence of the integral

$$\int_0 \xi M\Big(\frac{\omega(\xi)}{C}\Big)\, d\xi < \infty \tag{3.11}$$

with some constant C. If M satisfies the Δ_2-condition, the choice of C is of no significance since, in this case, $K_M = L_M$.

Example 3.1. The function $M \in \mathcal{K}_1$, i.e., it satisfies (1.7), for example, $M(s) = \exp(e^s)$ (this function M is close to the lower boundary of the class $\mathcal{K}_1$). We set $\omega(\xi) = \ln\alpha + \ln\ln(1/\xi)$. It is easy to verify that condition (3.11) is satisfied for $\alpha < 2$. In this case,

$$\int_0^r \xi\omega(\xi)\,d\xi = \frac{1}{2}\Big(-\operatorname{Ei}\Big(-2\ln\frac{1}{r}\Big) + r^2\Big(\ln\alpha + \ln\ln\frac{1}{r}\Big)\Big) \sim \frac{1}{2}\,r^2\ln\ln\frac{1}{r},$$

where $\operatorname{Ei}(z) = \int\limits_{-\infty}^{z} \frac{e^\eta}{\eta}\,d\eta$, i.e., after multiplication by 2, the field (3.10) becomes

$$\boldsymbol{u} \sim \Big(\ln\ln\frac{1}{r}\Big)\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}. \tag{3.12}$$

Example 3.2. The function M satisfies the condition (3.2), for example, $M(s) = s\ln^\beta s$, where $\beta > 1/2$. We set $\omega(\xi) = \xi^{-2}\ln^{-\alpha-\beta}(1/\xi)$. It is easy to verify that (3.11) is satisfied for $\alpha > 1$. In this case,

$$\int_0^r \xi\omega(\xi)\,d\xi = \frac{\ln^{1-\alpha-\beta}(1/r)}{\alpha+\beta-1}.$$

After multiplication by the constant, the field (3.10) becomes

$$\boldsymbol{u} \sim r^{-2}\Big(\ln^{-\gamma}\frac{1}{r}\Big)\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}, \qquad \gamma > \frac{1}{2}. \tag{3.13}$$

Having such fields $\boldsymbol{u}$, one can move the singularity from zero to other points of the domain and combine the fields with each other and with suitable smooth solenoidal fields [in such a manner that the sum satisfies condition (4)]. As a result, we obtain the initial fields $\boldsymbol{v}_0$ with point singularities of the classes considered in the present work [i.e., satisfying (3.3) with functions M of the corresponding classes].

Conclusions. Thus, some of the results of the investigation can be formulated as follows: if the singularities of the initial velocity are not worse than the singularity (3.13), it is possible to construct a solution (with finite kinetic energy) of problem (1)–(4), which, for all t, preserves regularity not worse than the initial one; if the singularities of the initial velocity are not worse than the singularity (3.12), the corresponding solution for all t belongs to the same class, and, in this class, the uniqueness of the solution is guaranteed.

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